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An algorithm for solving multi-term diffusion-wave equations of fractional order

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ABSTRACT

In this paper an algorithm, based on a new modified homotopy perturbation method (MHPM), is presented to obtain approximate solutions of multi-term diffusion-wave equations of fractional order. To illustrate the method some examples are provided. The results show the simplicity and the efficiency of the algorithm.

1. Introduction

The theory of derivatives of non-integer order goes back to the Leibniz's note in which the meaning of the derivative of order one half is discussed. In recent years, fractional differential equations (FDEs) have successfully modeled many phenomena in viscoelastic materials, polymer physics [1–3], fluid mechanics, biology, chemistry, acoustics, psychology [4], and other areas of science. For further applications, see [5] and references therein. Many powerful methods such as the Laplace transform method [2,6], the iterative method [7], the Fourier transform method [8], the operational method [9], the Adomian decomposition method [10–15], variational iteration method (VIM) [15], the homotopy analysis method (HAM) [16,17], and the homotopy perturbation method (HPM) [18–21] have been presented to solve FDEs. HPM has been applied with great success to obtain approximate solutions of a large variety of linear and nonlinear problems in ordinary differential equations (ODEs) [22,23], partial differential equations (PDEs) [24–26], and integral equations [27,28]. In addition, some modifications for HPM have been suggested [29–31]. The present paper is devoted to apply new modified HPM (MHPM) for solving some models of diffusion-wave equations of fractional order. We present an algorithm for solving multi-term diffusion-wave equations of fractional order based on new modified HPM.

This paper is organized as follows. In Section 2, some basic definitions, mathematical preliminaries of the fractional calculus theory, and formulation of the problem are introduced. Subsequently, in Section 3, the analysis of the method is presented. In Section 4, some examples are provided. Finally, in Section 5, conclusions are presented.

2. Preliminaries, notations and formulation of the problem

There are various definitions of fractional differentiation and integration, such as Grunwald–Letnikov's definition, Riemann–Liouville's definition, Caputo's definition and generalized function approach. In this paper the Caputo's definition of fractional differentiation is used.

Definition 1. A real function $f(x), x > 0$, is said to be in the space C_μ $(\mu \in R)$ if it can be written as $f(x) = x^p f_1(x)$ for some $p > \mu$ where $f_1(x)$ is a continuous function in $[0, \infty)$, and it is said to be in the space C_μ^m iff $f^{(m)} \in C_\mu,\ m \in \mathbb{N}$.

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Definition 2. Caputo's definition of the fractional order derivative is defined as

$$D^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\int_0^x\frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha+1-n}}\mathrm{d}\tau,\quad (n-1<\alpha\leq n,\ n\in\mathbb{N}),\tag{1}$$

where the parameter α is the order of the derivative, and $\Gamma(.)$ is the well-known Euler's Gamma function.

Caputo's derivative has the following properties [32]:

$$\begin{cases}D^{\alpha}C=0, & (C \text{ is a constant}),\\ D^{\alpha}x^{\beta}=0 & \text{for } \beta\leq\alpha-1,\\ D^{\alpha}x^{\beta}=\dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha} & \text{for } \beta>\alpha-1.\end{cases}\tag{2}$$

Moreover, Caputo's fractional differentiation is a linear operator and satisfies the so-called Leibnitz rule, i.e.

$$\begin{cases}D^{\alpha}[\lambda f(x)+\mu g(x)]=\lambda D^{\alpha}f(x)+\mu D^{\alpha}g(x),\\ D^{\alpha}[f(x)g(x)]=\displaystyle\sum_{k=0}^{\infty}\binom{\alpha}{k}f^{(k)}(x)D^{\alpha-k}g(x).\end{cases}\tag{3}$$

Definition 3. The Riemann–Liouville fractional integral operator of order $\alpha\geq 0$, of a function $f\in C_{\mu},\ \mu\geq -1$, is defined as

$$J^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}f(t)\mathrm{d}t,\quad \alpha>0,\ x>0.\tag{4}$$

Riemann–Liouville fractional integral operator satisfies the following properties for $f\in C_{\mu},\ \mu\geq -1,\ \alpha,\beta\geq 0,\ \gamma\geq -1$ [2]:

$$\begin{cases}J^{0}f(x)=f(x),\\ J^{\alpha}x^{\gamma}=\dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma},\\ J^{\alpha}J^{\beta}f(x)=J^{\alpha+\beta}f(x),\\ J^{\alpha}J^{\beta}f(x)=J^{\beta}J^{\alpha}f(x),\end{cases}\tag{5}$$

and

$$\begin{cases}D^{\alpha}J^{\alpha}f(x)=f(x),\\ J^{\alpha}D^{\alpha}f(x)=f(x)-\displaystyle\sum_{i=0}^{m-1}f^{(i)}(0^{+})\frac{x^i}{i!},\quad x>0,\end{cases}\tag{6}$$

where $m-1<\alpha\leq m,\ m\in\mathbb{N}$ and $f\in C_{\mu}^{m}$.

Definition 4. A two-parameter Mittag-Leffler function is defined by the following series

$$E_{\alpha,\beta}(t)=\sum_{j=0}^{\infty}\frac{t^j}{\Gamma(\alpha j+\beta)}.\tag{7}$$

Consequently, the kth derivative of the Mittag-Leffler function is obtained as

$$E_{\alpha,\beta}^{(k)}(t)=\frac{\partial^k}{\partial x^k}E_{\alpha,\beta}(t)=k!\sum_{j=k}^{\infty}\binom{j}{k}\frac{t^{j-k}}{\Gamma(\alpha j+\beta)}.\tag{8}$$

Now, we consider the following fractional order diffusion-wave problem

$$P(D)u(\bar{x},t)=\sum_{i=1}^{n}N_i(\bar{x},t)\frac{\partial^2u(\bar{x},t)}{\partial x_i^2}+\Phi(\bar{x},t)u^m(\bar{x},t),\tag{9}$$

subject to the initial conditions

$$\begin{cases}u(\bar{x},0)=f(\bar{x}),\\ u_t(\bar{x},t)=g(\bar{x}),\end{cases}\tag{10}$$

where

$$P(D) = D_t^{s_1} - \sum_{j=2}^{r} \lambda_j D_t^{s_j}, \tag{11}$$

and $0 < s_r < s_{r-1} < \cdots < s_{r-k} < 1 < s_{r-(k+1)} < \cdots < s_2 < s_1 < 2$. In the next section, we introduce an algorithm to solve problem (9)–(11).

3. Analysis of the MHPM

First of all, for convenience of the reader, a short review of HPM is presented, then a new MHPM for solving multi-term diffusion-wave equations is presented.

He presented a homotopy perturbation technique based on the introduction of a homotopy and an artificial parameter for the solution of algebraic and ODEs [33]. To describe HPM, we consider the following nonlinear differential equation

$$A(\upsilon) - f(r) = 0, \quad r \in \Omega, \tag{12}$$

with the boundary conditions

$$B\left(\upsilon, \frac{\partial \upsilon}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{13}$$

where A is a differential operator, B is a boundary operator, $f(r)$ is a known analytic function, and Γ is the boundary of the domain Ω. The operator A can be divided into two parts L and N, where L is a linear operator and N is a nonlinear operator. Therefore, Eq. (12) can be rewritten as

$$L(\upsilon) + N(\upsilon) - f(r) = 0. \tag{14}$$

J.H. He constructs the following homotopy $\upsilon(r, p) : \Omega \times [0, 1] \longrightarrow R$ which satisfies

$$\begin{aligned} H(\upsilon, p) &= (1-p)[L(\upsilon) - L(u_0)] + p[A(\upsilon) - f(r)] \\ &= L(\upsilon) - (1-p)L(u_0) + p[N(\upsilon) - f(r)] = 0, \quad p \in [0, 1],\ r \in \Omega, \end{aligned} \tag{15}$$

where $p \in [0, 1]$ is an embedding parameter, and u_0 is an initial approximation which satisfies the boundary conditions. How to choose the initial approximation is very important in this method [34]. Obviously we have

$$\begin{cases} H(\upsilon, 0) = L(\upsilon) - L(u_0) = 0, \\ H(\upsilon, 1) = A(\upsilon) - f(r) = 0. \end{cases}$$

Changing the process of p from zero to unity is just that of $\upsilon(r, p)$ from $u_0(r)$ to $\upsilon(r)$. We assume that the solution of Eq. (15) can be written as a power series in p, i.e.

$$\upsilon = \upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + \cdots = \sum_{i=0}^{\infty} p^i \upsilon_i. \tag{16}$$

By setting (16) in (15) and equating the terms by the same power in p, a successive procedure to determine υ_i is obtained. Finally, by setting $p = 1$ in (16), the solution of (12) is obtained. In practice, if the series could not be recognized as the series of a known function, the first few terms will be taken as an approximation to the solution.

Now we apply HPM to solve problem (9)–(11). By using (11), (9) becomes

$$D_t^{s_1} u(\bar{x}, t) = \sum_{j=2}^{r} \lambda_j D_t^{s_j} u(\bar{x}, t) + \sum_{i=1}^{n} N_i(\bar{x}, t) \frac{\partial^2 u(\bar{x}, t)}{\partial x_i^2} + \phi(\bar{x}, t) u^m(\bar{x}, t). \tag{17}$$

For this problem, the following homotopy have been applied [35,36]:

For $m = 0$:

$$\upsilon^{(c)} - \phi(\bar{x}, t) = p\left[\upsilon^{(c)} + \sum_{j=2}^{r} \lambda_j D_t^{s_j} \upsilon + \sum_{i=1}^{n} N_i(\bar{x}, t) \frac{\partial^2 \upsilon}{\partial x_i^2} - D_t^{s_1} \upsilon\right]. \tag{18}$$

For $m \in \mathbb{N}$:

$$\upsilon^{(c)} = p\left[\upsilon^{(c)} + \sum_{j=2}^{r} \lambda_j D_t^{s_j} \upsilon + \sum_{i=1}^{n} N_i(\bar{x}, t) \frac{\partial^2 \upsilon}{\partial x_i^2} + \phi(\bar{x}, t)\upsilon^m - D_t^{s_1} \upsilon\right], \tag{19}$$

where $c - 1 < s_1 \le c$, and $c \in \mathbb{N}$. In this paper, we apply the following new modified homotopy.

For $m=0$:

$$D_t^{s_1}\upsilon-\phi(\bar{x},t)=p\left[\sum_{j=2}^{r}\lambda_j D_t^{s_j}\upsilon+\sum_{i=1}^{n}N_i(\bar{x},t)\frac{\partial^2\upsilon}{\partial x_i^2}\right]. \tag{20}$$

For $m\in\mathbb{N}$:

$$D_t^{s_1}\upsilon=p\left[\sum_{j=2}^{r}\lambda_j D_t^{s_j}\upsilon+\sum_{i=1}^{n}N_i(\bar{x},t)\frac{\partial^2\upsilon}{\partial x_i^2}+\phi(\bar{x},t)\upsilon^m\right]. \tag{21}$$

In this case to obtain υ_0 (i.e. initial approximation), it is sufficient to consider the following approach. By applying $D_t^{-s_1}$ on both sides of (17), we have

$$u(\bar{x},t)-f(\bar{x})-tg(\bar{x})=\sum_{j=2}^{r}\lambda_j D_t^{-s_1}D_t^{s_j}u(\bar{x},t)+\sum_{i=1}^{n}D_t^{-s_1}\left[N_i(\bar{x},t)\frac{\partial^2 u(\bar{x},t)}{\partial x_i^2}\right]+D_t^{-s_1}[\phi(\bar{x},t)u^m(\bar{x},t)],\quad m\in\mathbb{N}.$$

Therefore we obtain

$$\begin{aligned}u(\bar{x},t)-f(\bar{x})-tg(\bar{x})&=\sum_{j=2}^{r-(k+1)}\lambda_j D_t^{-(s_1-s_j)}(D_t^{-s_j}D_t^{s_j}u(\bar{x},t))+\sum_{j=r-k}^{r}\lambda_j D_t^{-(s_1-s_j)}(D_t^{-s_j}D_t^{s_j}u(\bar{x},t))\\&\quad+\sum_{i=1}^{n}D_t^{-s_1}\left[N_i(\bar{x},t)\frac{\partial^2 u(\bar{x},t)}{\partial x_i^2}\right]+D_t^{-s_1}[\phi(\bar{x},t)u^m(\bar{x},t)],\quad m\in\mathbb{N}.\end{aligned}$$

By applying (6) we have

$$\begin{aligned}u(\bar{x},t)-f(\bar{x})-tg(\bar{x})&=\sum_{j=2}^{r-(k+1)}\lambda_j D_t^{-(s_1-s_j)}(u(\bar{x},t)-f(\bar{x})-tg(\bar{x}))+\sum_{j=r-k}^{r}\lambda_j D_t^{-(s_1-s_j)}(u(\bar{x},t)-f(\bar{x}))\\&\quad+\sum_{i=1}^{n}D_t^{-s_1}\left[N_i(\bar{x},t)\frac{\partial^2 u(\bar{x},t)}{\partial x_i^2}\right]+D_t^{-s_1}[\phi(\bar{x},t)u^m(\bar{x},t)],\quad m\in\mathbb{N}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}u(\bar{x},t)-f(\bar{x})-tg(\bar{x})&=-f(\bar{x})\sum_{j=2}^{r-(k+1)}\lambda_j D_t^{-(s_1-s_j)}(1)-f(\bar{x})\sum_{j=r-k}^{r}\lambda_j D_t^{-(s_1-s_j)}(1)\\&\quad-g(\bar{x})\sum_{j=2}^{r-(k+1)}\lambda_j D_t^{-(s_1-s_j)}(t)+\sum_{j=2}^{r}\lambda_j D_t^{-(s_1-s_j)}u(\bar{x},t)\\&\quad+\sum_{i=1}^{n}D_t^{-s_1}\left[N_i(\bar{x},t)\frac{\partial^2 u(\bar{x},t)}{\partial x_i^2}\right]+D_t^{-s_1}[\phi(\bar{x},t)u^m(\bar{x},t)],\quad m\in\mathbb{N}.\end{aligned} \tag{22}$$

Relations (5) and (6), simplify (22) as

$$\begin{aligned}u(\bar{x},t)&=-f(\bar{x})\left[\sum_{j=1}^{r}\lambda_j\frac{t^{s_1-s_j}}{\Gamma(1+s_1-s_j)}\right]-g(\bar{x})\left[\sum_{j=1}^{r-(k+1)}\lambda_j\frac{t^{1+s_1-s_j}}{\Gamma(2+s_1-s_j)}\right]\\&\quad+\sum_{j=2}^{r}\lambda_j D_t^{-(s_1-s_j)}u(\bar{x},t)+\sum_{i=1}^{n}D_t^{-s_1}\left[N_i(\bar{x},t)\frac{\partial^2 u(\bar{x},t)}{\partial x_i^2}\right]+D_t^{-s_1}[\phi(\bar{x},t)u^m(\bar{x},t)],\end{aligned} \tag{23}$$

where $\lambda_1=-1$.

Therefore, for $m=0$ we set

$$\upsilon_0=-f(\bar{x})\left[\sum_{j=1}^{r}\lambda_j\frac{t^{s_1-s_j}}{\Gamma(1+s_1-s_j)}\right]-g(\bar{x})\left[\sum_{j=1}^{r-(k+1)}\lambda_j\frac{t^{1+s_1-s_j}}{\Gamma(2+s_1-s_j)}\right]+D_t^{-s_1}\phi(\bar{x},t). \tag{24}$$

Similarly for $m\in\mathbb{N}$ we set

$$\upsilon_0=-f(\bar{x})\left[\sum_{j=1}^{r}\lambda_j\frac{t^{s_1-s_j}}{\Gamma(1+s_1-s_j)}\right]-g(\bar{x})\left[\sum_{j=1}^{r-(k+1)}\lambda_j\frac{t^{1+s_1-s_j}}{\Gamma(2+s_1-s_j)}\right]. \tag{25}$$

These initial approximate solutions are applied in the Adomian decomposition method [14]. By substituting (16) into (20) or (21), the unknown terms υ_i is obtained. The following equations, generate all terms υ_i for $m=0,1,2$.

- *Case* 1. For $m = 0$, we obtain

coefficient of p^0:

$$\upsilon_0 = -f(\bar{x})\left[\sum_{j=1}^{r}\lambda_j\frac{t^{s_1-s_j}}{\Gamma(1+s_1-s_j)}\right] - g(\bar{x})\left[\sum_{j=1}^{r-(k+1)}\lambda_j\frac{t^{1+s_1-s_j}}{\Gamma(2+s_1-s_j)}\right] + D_t^{-s_1}\phi(\bar{x}, t),$$

coefficient of p^1:

$$\upsilon_1 = \sum_{j=2}^{r}\lambda_j D_t^{-(s_1-s_j)}\upsilon_0 + \sum_{i=1}^{n} D_t^{-s_1}\left[N_i(\bar{x}, t)\frac{\partial^2\upsilon_0}{\partial x_i^2}\right],$$

$\vdots$

coefficient of p^k:

$$\upsilon_k = \sum_{j=2}^{r}\lambda_j D_t^{-(s_1-s_j)}\upsilon_{k-1} + \sum_{i=1}^{n} D_t^{-s_1}\left[N_i(\bar{x}, t)\frac{\partial^2\upsilon_{k-1}}{\partial x_i^2}\right].$$

- *Case* 2. For $m = 1$, we obtain

coefficient of p^0:

$$\upsilon_0 = -f(\bar{x})\left[\sum_{j=1}^{r}\lambda_j\frac{t^{s_1-s_j}}{\Gamma(1+s_1-s_j)}\right] - g(\bar{x})\left[\sum_{j=1}^{r-(k+1)}\lambda_j\frac{t^{1+s_1-s_j}}{\Gamma(2+s_1-s_j)}\right],$$

coefficient of p^1:

$$\upsilon_1 = \sum_{j=2}^{r}\lambda_j D_t^{-(s_1-s_j)}\upsilon_0 + \sum_{i=1}^{n} D_t^{-s_1}\left[N_i(\bar{x}, t)\frac{\partial^2\upsilon_0}{\partial x_i^2}\right] + D_t^{-s_1}[\phi(\bar{x}, t)\upsilon_0],$$

$\vdots$

coefficient of p^k:

$$\upsilon_k = \sum_{j=2}^{r}\lambda_j D_t^{-(s_1-s_j)}\upsilon_{k-1} + \sum_{i=1}^{n} D_t^{-s_1}\left[N_i(\bar{x}, t)\frac{\partial^2\upsilon_{k-1}}{\partial x_i^2}\right] + D_t^{-s_1}[\phi(\bar{x}, t)\upsilon_{k-1}].$$

- *Case* 3. For $m = 2$, we obtain

coefficient of p^0:

$$\upsilon_0 = -f(\bar{x})\left[\sum_{j=1}^{r}\lambda_j\frac{t^{s_1-s_j}}{\Gamma(1+s_1-s_j)}\right] - g(\bar{x})\left[\sum_{j=1}^{r-(k+1)}\lambda_j\frac{t^{1+s_1-s_j}}{\Gamma(2+s_1-s_j)}\right],$$

coefficient of p^1:

$$\upsilon_1 = \sum_{j=2}^{r}\lambda_j D_t^{-(s_1-s_j)}\upsilon_0 + \sum_{i=1}^{n} D_t^{-s_1}\left[N_i(\bar{x}, t)\frac{\partial^2\upsilon_0}{\partial x_i^2}\right] + D_t^{-s_1}[\phi(\bar{x}, t)\upsilon_0^2],$$

coefficient of p^2:

$$\upsilon_2 = \sum_{j=2}^{r}\lambda_j D_t^{-(s_1-s_j)}\upsilon_1 + \sum_{i=1}^{n} D_t^{-s_1}\left[N_i(\bar{x}, t)\frac{\partial^2\upsilon_1}{\partial x_i^2}\right] + 2D_t^{-s_1}[\phi(\bar{x}, t)(\upsilon_0\upsilon_1)],$$

coefficient of p^3:

$$\upsilon_3 = \sum_{j=2}^{r}\lambda_j D_t^{-(s_1-s_j)}\upsilon_2 + \sum_{i=1}^{n} D_t^{-s_1}\left[N_i(\bar{x}, t)\frac{\partial^2\upsilon_2}{\partial x_i^2}\right] + D_t^{-s_1}[\phi(\bar{x}, t)(\upsilon_1^2 + 2\upsilon_0\upsilon_2)],$$

$\vdots$

coefficient of p^k:

$$\upsilon_k = \sum_{j=2}^{r}\lambda_j D_t^{-(s_1-s_j)}\upsilon_{k-1} + \sum_{i=1}^{n} D_t^{-s_1}\left[N_i(\bar{x}, t)\frac{\partial^2\upsilon_{k-1}}{\partial x_i^2}\right] + D_t^{-s_1}\left[\phi(\bar{x}, t)\frac{1}{(k-1)!}\left[\frac{\mathrm{d}^{k-1}}{\mathrm{d}p^{k-1}}\left(\sum_{i=0}^{\infty}p^i\upsilon_i\right)^2\right]_{p=0}\right].$$

In general for arbitrary $m, k \in \mathbb{N}$ we have

$$\upsilon_k = \sum_{j=2}^{r} \lambda_j D_t^{-(s_1-s_j)} \upsilon_{k-1} + \sum_{i=1}^{n} D_t^{-s_1} \left[N_i(\bar{x}, t) \frac{\partial^2 \upsilon_{k-1}}{\partial x_i^2} \right] + D_t^{-s_1} \left[\phi(\bar{x}, t) \frac{1}{(k-1)!} \left[\frac{\mathrm{d}^{k-1}}{\mathrm{d}p^{k-1}} \left(\sum_{i=0}^{\infty} p^i \upsilon_i \right)^m \right]_{p=0} \right]. \tag{26}$$

4. Illustrative examples

Example 1. Consider the following linear fractional differential equation [37]:

$$\frac{\mathrm{d}^\alpha y}{\mathrm{d}t^\alpha} + w^{\alpha-\beta} \frac{\mathrm{d}^\beta y}{\mathrm{d}t^\beta} = 0, \quad 1 < \alpha \leq 2,\ 0 \leq \beta < 1, \tag{27}$$

subject to initial conditions

$$\begin{cases} y(0) = 0, \\ y'(0) = 1, \end{cases} \tag{28}$$

and w is a constant parameter. For $\beta = 0$, this equation describes a simple harmonic fractional oscillator and has been solved in [38] by using the Laplace transform method and the Mittag-Leffler function. By using (20) we obtain

$$\begin{cases} \upsilon_0 = t, \\ \upsilon_1 = -w^{\alpha-\beta} \dfrac{t^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}, \\ \upsilon_2 = w^{2(\alpha-\beta)} \dfrac{t^{2(\alpha-\beta)+1}}{\Gamma(2(\alpha-\beta)+2)}, \\ \upsilon_3 = -w^{3(\alpha-\beta)} \dfrac{t^{3(\alpha-\beta)+1}}{\Gamma(3(\alpha-\beta)+2)}, \\ \vdots \\ \upsilon_k = (-1)^k w^{k(\alpha-\beta)} \dfrac{t^{k(\alpha-\beta)+1}}{\Gamma(k(\alpha-\beta)+2)}. \end{cases}$$

Hence we obtain

$$y(t) = \sum_{k=0}^{\infty} (-1)^k w^{k(\alpha-\beta)} \frac{t^{k(\alpha-\beta)+1}}{\Gamma(k(\alpha-\beta)+2)} = tE_{\alpha-\beta,2}(-(wt)^{\alpha-\beta}). \tag{29}$$

It is clear that in Eq. (29) when $\alpha \longrightarrow 2$ and $\beta = 0$, the exact solution of simple harmonic oscillator is obtained.

Example 2. Consider the following nonlinear fractional differential equation [14]:

$$(D_t^{\frac{3}{2}} - \lambda D_t^{\frac{1}{2}})u + u_{xx} + u^2 = 0, \quad \infty < x < \infty,\ t > 0, \tag{30}$$

with initial conditions

$$u(x, 0) = x, \qquad u_t(x, 0) = \sin x. \tag{31}$$

Similarly by using (21) and (25) we obtain

$$\begin{cases} \upsilon_0 = x(1-t) + t\sin x, \\ \upsilon_1 = tx - 0.5t^2 x + 0.5t^2 \sin x + 0.300901 t^{2.5} \sin x - t^{1.5}((0.752253 + (-0.601802 + 0.171943t)t)x^2 \\ \quad + t\sin x((0.601802 - 0.343887t)x + 0.171943t\sin x)), \\ \vdots \end{cases}$$

In practice, the first few terms can be taken as an approximation to the solution (i.e. $u \approx \upsilon_0 + \upsilon_1 + \cdots + \upsilon_n$).

5. Conclusion

In this paper, we proposed the new modified homotopy perturbation method for solving multi-term diffusion-wave equations of fractional order. In addition, the explicit formulas to obtain the unknown terms υ_i are presented. Finally, we would like to emphasize that our new method with introduced initial approximations υ_0 (i.e. (24) and (25)) is coincident to the Adomian decomposition method.

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